ONE-DIMENSIONAL CONVECTIVE HEATING WITH A TIME-DEPENDENT HEAT-TRANSFER COEFFICIENT

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The problem of the symmetric convective heating of a plate, cylinder, and sphere with a time-dependent heat-transfer coefficient is solved by the method of averaging functional corrections.

The analytic study of the convective heating process with a time-varying heat-transfer coefficient requires the solution of the Fourier equation

$$\frac{\partial T}{\partial t}=\frac{a}{x^m}\frac{\partial}{\partial x}\left[x^m\frac{\partial T}{\partial x}\right], \tag{1}$$

describing the symmetric heating of a plate (m = 0), cylinder (m = 1), or sphere (m = 2) with a boundary condition of the third kind corresponding to Newton's law

$$\lambda\left.\frac{\partial T}{\partial x}\right|_{x=R}=\alpha(t)\left[T_a-T_s(t)\right]. \tag{2}$$

It is assumed that the coefficients a and λ, the ambient temperature T_a, and the initial temperature T_0 are constants:

$$T(x,\ 0)=T_0=\text{const.} \tag{3}$$

We shall stipulate a second boundary condition later. If we introduce the dimensionless variables

$$\xi=\frac{x}{R};\quad \tau=\frac{at}{R^2}=\text{Fo};\quad \text{Bi}(\tau)=\frac{\alpha(t)R}{\lambda};\quad \theta(\xi,\ \tau)=\frac{T(x,\ t)}{T_a} \tag{4}$$

and, following [1], the new function

$$u(\xi,\ \tau)=\ln\left[1-\theta(\xi,\ \tau)\right], \tag{5}$$

Eqs. (1)-(3) are transformed to

$$\frac{\partial u}{\partial\tau}=\frac{\partial^2 u}{\partial\xi^2}+\frac{m}{\xi}\frac{\partial u}{\partial\xi}+\left(\frac{\partial u}{\partial\xi}\right)^2; \tag{6}$$

$$\left.\frac{\partial u}{\partial\xi}\right|_{\xi=1}=-\text{Bi}(\tau); \tag{7}$$

$$u(\xi,\ 0)=\ln(1-\theta_0)=u_0. \tag{8}$$

The term $(\partial u/\partial\xi)^2$ of Eq. (6) is omitted in [1], thus restricting the discussion to thin bodies.

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In accord with the conventional [2] engineering model of the heating process we consider two successive stages: inertial (warming up of the body) and regular (heating over the whole cross section). The problem is solved by the method of averaging functional corrections [3-5].

Inertial Heating ($0 \le \tau \le \tau_0$). We assume that the temperature distributions at the boundary between the heated and unheated zones are joined. In this case there must be added to Eq. (6) with initial condition (8) and boundary condition (7) the joining conditions

$$u(\xi, \tau)|_{\xi=\rho(\tau)} = u_0, \tag{9}$$

$$\left.\frac{\partial u}{\partial \xi}\right|_{\xi=\rho(\tau)} = 0; \tag{10}$$

where

$$\rho(\tau) = \frac{r(t)}{R}; \tag{11}$$

r(t) is the width of the unperturbed zone of the cross section of the body, i.e., the distance from the center of the cross section to the front of the moving thermal perturbation.

As in [4, 5] we set

$$\frac{\partial^2 u_1}{\partial \xi^2} = f_1(\tau), \tag{12}$$

where

$$f_1(\tau) = \frac{1}{1-\rho(\tau)} \int\limits_{\rho(\tau)}^{1} \left[\frac{\partial u_1}{\partial \tau} - \left(\frac{\partial u}{\partial \xi}\right)^2 - \frac{m}{\xi}\,\frac{\partial u_1}{\partial \xi}\right] d\xi. \tag{13}$$

Integrating (12) twice with respect to ξ and using (9) and (10) we have

$$u_1(\xi, \tau) = u_0 - \frac{\mathrm{Bi}(\tau)}{2[1-\rho(\tau)]}\,[\xi - \rho(\tau)]^2 \quad (\rho \leqslant \xi \leqslant 1). \tag{14}$$

From (7) we find

$$f_1(\tau) = \frac{\mathrm{Bi}(\tau)}{1-\rho(\tau)}. \tag{15}$$

Substituting (14) and (15) into (13) and making a number of transformations we find the following differential equation for the remaining unknown function $\rho(\tau)$:

$$\frac{d}{d\tau}\{\mathrm{Bi}(\tau)[1-\rho(\tau)]^2\} + 2\,\mathrm{Bi}^2(\tau)[1-\rho(\tau)] = 6(m+1)\,\mathrm{Bi}(\tau). \tag{16}$$

Here by analogy with [5] we have omitted the terms containing the factor $\rho(\tau)\ln\rho(\tau)$.

Introducing the notation

$$g(\tau) = \sqrt{\mathrm{Bi}(\tau)}\,\beta(\tau), \tag{17}$$

$$\beta(\tau) = 1 - \rho(\tau), \tag{18}$$

we rewrite Eq. (16) in the form

$$g(\tau)\frac{dg}{d\tau} + \mathrm{Bi}(\tau)\sqrt{\mathrm{Bi}(\tau)}\,g(\tau) = 3(m+1)\,\mathrm{Bi}(\tau). \tag{19}$$

Equation (19) is a special case of the well-known Abel's equation of the second kind [6].

Setting

$$g(\tau) = v(\tau) + \Phi(\tau), \tag{20}$$

$$\Phi(\tau) = -\int \mathrm{Bi}^{3/2}(\tau)\,d\tau, \tag{21}$$

we obtain

$$[v(\tau)+\Phi(\tau)]\,\dot{v}(\tau)=3(m+1)\,\mathrm{Bi}(\tau), \tag{22}$$

which, according to [6], cannot be integrated in closed form. We obtain an approximate solution of (22) by setting the additive function $\Phi(\tau)$ on the left-hand side of (22) equal to some average value Φ_*. This is admissible because of the transient nature of the inertial heating process.

In this case the integral of Eq. (22) is

$$\frac{v^2}{2}+\Phi_* v=3(m+1)\int \mathrm{Bi}(\tau)\,d\tau+C. \tag{23}$$

By using (20) and (17) Eq. (23) is reduced to the following expression for the depth of penetration of the thermal perturbation

$$\beta(\tau)=\sqrt{\frac{6(m+1)}{\mathrm{Bi}(\tau)}\left[\int \mathrm{Bi}(\tau)\,d\tau+D\right]}, \tag{24}$$

where the constant D is determined from the obvious initial condition

$$\beta(0)=0 \tag{25}$$

in each specific case the function $\mathrm{Bi}(\tau)$ is given. If we set $\mathrm{Bi}(\tau)$ = const in (24)

$$1-\rho(\tau)=\beta(\tau)=\sqrt{6(m+1)\tau}, \tag{26}$$

which agrees with the results in [5].

Thus the problem of the inertial stage of the heating has been solved.

Returning to the original notation we can write finally

$$\frac{1-\theta_1(\xi,\tau)}{1-\theta_0}=\exp\left\{-\frac{\mathrm{Bi}(\tau)\,\beta(\tau)}{2}\left[1-\frac{1-\xi}{\beta(\tau)}\right]^2\right\}. \tag{27}$$

If we set $\xi = 1$ in (27) we obtain the surface temperature of the body

$$\theta_{1s}(\tau)=1-(1-\theta_0)\exp\left\{-\frac{\mathrm{Bi}(\tau)\,\beta(\tau)}{2}\right\}. \tag{28}$$

Introducing the idea of the Biot number for a heated layer with a time-varying thickness $\beta(\tau)$

$$\widetilde{\mathrm{Bi}}(\tau)=\frac{\alpha(t)\,l(t)}{\lambda}=\mathrm{Bi}(\tau)\,\beta(\tau), \tag{29}$$

Eq. (28) can be written in a form analogous to the corresponding formula of [5]:

$$\theta_{1s}(\tau)=1-(1-\theta_0)\exp\left[\frac{\widetilde{\mathrm{Bi}}(\tau)}{2}\right]. \tag{30}$$

The duration of the first stage of the heating can be determined for any given $\mathrm{Bi}(\tau)$ by setting $\tau = \tau_0$ and $\beta(\tau_0) = 1$ in (24).

Regular Heating ($\tau_0 \le \tau < \infty$). To investigate regular heating Eq. (6) must be solved with boundary condition (7) and the condition for the symmetry of the heating

$$\left.\frac{\partial u}{\partial \xi}\right|_{\xi=0}=0. \tag{31}$$

We set

$$\frac{\partial^2 u_2}{\partial \xi^2}=f_2(\tau), \tag{32}$$

$$f_2(\tau)=\int_0^1\left[\frac{\partial u_2}{\partial \tau}-\left(\frac{\partial u_2}{\partial \xi}\right)^2-\frac{m}{\xi}\,\frac{\partial u_2}{\partial \xi}\right]d\xi. \tag{33}$$

Integrating (32) twice with respect to ξ and using boundary conditions (7) and (31) we have

$$u_2(\xi, \tau) = u_{2S}(\tau) + \frac{\mathrm{Bi}(\tau)}{2}(1-\xi^2). \tag{34}$$

After substituting (34) into (33) and integrating we obtain the following differential equation for the unknown function $u_{2S}(\tau)$:

$$\frac{d}{d\tau}\left[u_{2S}(\tau) + \frac{1}{3}\mathrm{Bi}(\tau)\right] = \frac{\mathrm{Bi}^2(\tau)}{3} - (m+1)\mathrm{Bi}(\tau), \tag{35}$$

which has the solution

$$u_{2S}(\tau) = \int_{\tau_0}^{\tau}\left[\frac{\mathrm{Bi}^2(\tau)}{3} - (m+1)\mathrm{Bi}(\tau)\right]d\tau - \frac{1}{3}\mathrm{Bi}(\tau) + C, \tag{36}$$

where the integration constant C is determined from the initial condition

$$u_{2S}(\tau_0) = u_{1S}(\tau_0) = u_0 - \frac{\mathrm{Bi}(\tau_0)}{2}. \tag{37}$$

Setting $\tau = \tau_0$ in (36) and using (37) we find

$$C = u_0 - \frac{\mathrm{Bi}(\tau_0)}{6}. \tag{38}$$

If we introduce the notation

$$\varphi(\tau) = \frac{\mathrm{Bi}(\tau_0)}{6} + \frac{\mathrm{Bi}(\tau)}{3} - \int_{\tau_0}^{\tau}\left[\frac{\mathrm{Bi}^2(\tau)}{3} - (m+1)\mathrm{Bi}(\tau)\right]d\tau, \tag{39}$$

we obtain finally

$$u_2(\xi, \tau) = u_0 - \varphi(\tau) + \frac{\mathrm{Bi}(\tau)}{2}(1-\xi^2). \tag{40}$$

Returning to the original notation we have

$$\frac{1-\theta_2(\xi_1\tau)}{1-\theta_0} = \exp\left\{-\varphi(\tau) + \frac{\mathrm{Bi}(\tau)}{2}(1-\xi^2)\right\}. \tag{41}$$

In heat engineering calculations one is generally interested in the temperatures of the surface of the body and at the center of its cross section

$$\theta_{2S}(\tau) = 1 - (1-\theta_0)e^{-\varphi(\tau)}, \tag{42}$$

$$\theta_{2C}(\tau) = 1 - (1-\theta_0)e^{-\varphi(\tau)+\frac{\mathrm{Bi}(\tau)}{2}}. \tag{43}$$

Thus the problem posed has been solved for an arbitrary continuous time-varying heat-transfer coefficient $\alpha(t)$.

Analysis of a number of papers devoted to the problem under study shows that in most of them the simplest laws of variation of the heat-transfer coefficient have been assumed. Various methods have been used: operational [7, 8, 11], variational [9], finite integral transforms [10], etc. All these methods lead to expressions which are rather complicated for practical use.

Only in [1] is the problem under study illustrated with a numerical example. In the expression for the Biot number

$$\mathrm{Bi}(\tau) = \mathrm{Bi}_0 - \mathrm{Bi}_1 e^{-\tau} \tag{44}$$

it is assumed that

$$\mathrm{Bi}_0 = 1.2; \quad \mathrm{Bi}_1 = 1.0. \tag{45}$$

TABLE 1. Temperature of the Surface $\theta_s(\tau)$ and in the Central Plane $\theta_c(\tau)$ at Various Times τ

τ	$\theta_s(\tau)$: from data in [12] $\theta_s^*(\tau)$	from data in [1] $\theta_s^\dagger(\tau)$	according to (28) and (42) $\theta_s(\tau)$	$\theta_s^* - \theta_s^\dagger$, %	$\theta_s^* - \theta_s$, %	$\theta_c(\tau)$: from data in [12] $\theta_c^*(\tau)$	from data in [1] $\theta_c^\dagger(\tau)$	according to (43) $\theta_c(\tau)$	$\theta_c^* - \theta_c^\dagger$, %	$(\theta_c^* - \theta_c^\dagger)$, %
0,00	—	—	0,336	—	—	—	—	0,336	—	—
0,05	—	—	0,375	—	—	—	—	0,336	—	—
0,10	—	—	0,400	—	—	—	—	0,336	—	—
0,15	—	—	0,425	—	—	—	—	0,336	—	—
0,212	—	—	0,451	—	—	—	—	0,336	—	—
0,50	0,526	0,556	0,545	—5,83	—3,49	0,400	0,397	0,390	+0,83	+2,56
1,00	0,678	0,714	0,680	—5,41	—0,30	0,538	0,569	0,516	—5,80	+4,36
1,50	0,769	0,814	0,778	—5,75	—1,16	0,662	0,702	0,638	—6,00	+3,78
2,00	0,847	0,898	0,848	—5,99	—0,12	0,770	0,815	0,738	—5,81	+4,34
2,50	0,883	0,928	0,894	—5,00	—1,23	0,833	0,878	0,814	—5,35	+2,34
3,00	0,921	0,954	0,925	—3,62	—0,43	0,878	0,928	0,868	—5,68	+1,10
3,50	0,947	0,976	0,948	—3,15	—0,11	0,917	0,956	0,907	—4,18	+1,10
4,00	0,964	0,990	0,964	—2,71	0,00	0,944	0,982	0,935	—4,04	+0,96
4,50	—	—	0,975	—	—	—	—	0,954	—	—
5,00	—	—	0,983	—	—	—	—	0,968	—	—

Only the regular heating of a plate is considered, with the initial relative temperature

$$\theta_0 = 0.336. \tag{46}$$

Using the variation of the Biot number given in (44) and the numerical values from (45) and (46) we determine the temperature of the surface $\theta_s(\tau)$ and in the central plane $\theta_c(\tau)$ of a plate and compare the results with data from [1] which in turn are compared with the more accurate solution obtained in [12] by the finite difference method.

Substituting (44) into (24) gives

$$\beta(\tau) = \sqrt{6(m+1)\,\frac{\mathrm{Bi}_0\tau + \mathrm{Bi}_1[e^{-\tau} - 1]}{\mathrm{Bi}_0 - \mathrm{Bi}_1 e^{-\tau}}}\,. \tag{47}$$

Assuming that the duration of the inertial heating stages τ_0 is relatively short we expand $\exp(-\tau_0)$ in a power series and retain second-order terms $\tau_0^k (k \le 2)$:

$$e^{-\tau_0} \simeq 1 - \tau_0 + \frac{\tau_0^2}{2}\,. \tag{48}$$

Using the fact that $\beta(\tau_0) = 1$ we obtain from (47)

$$\tau_0 = \frac{\tau_0^*}{1+\tau_0^*} \cdot \frac{\sqrt{1 + 2\tau_0^*\gamma + \gamma^2} - (1-\gamma)}{\gamma}\,, \tag{49}$$

where

$$\gamma = \frac{\mathrm{Bi}_1\tau_0^*}{\mathrm{Bi}_0 - \mathrm{Bi}_1}\,; \tag{50}$$

$$\tau_0^* = \frac{1}{6(m+1)}\,.$$

Equation (51) determines the duration of the inertial heating stage of a plate for α = const [5]. Analysis of Eq. (49) shows that the inertial heating proceeds more slowly when the Biot number varies according to (44) than when it is constant ($\tau_0 > \tau_0^*$).

For a plate (m = 0) with numbers from (45) we have from (49)

$$\tau_0 = 0.212. \tag{52}$$

We now calculate $\varphi(\tau)$. Substituting into (39), (44), (45), and (52) for m = 0 we obtain

$$\varphi(\tau) = 0.041 + 0.72\tau - 0.133e^{-\tau} + 0.167e^{-2\tau}. \tag{53}$$

The relative temperatures of the surface and the center of the cross section of the plate are now found from Eqs. (28), (42), and (43) using θ_0 from (46). The results are given in Table 1 which also lists the results from [1, 12].

A comparative analysis shows that the averaging of functional corrections, even in first approximation, is more successful than the procedure given in [1] in approximating the results of the finite difference method.

In addition the exceptional simplicity both of the working formulas and of the calculational procedure itself from the practical point of view distinguishes our method from most other known methods which lead to solutions in infinite series by using integral representations and special functions.

NOTATION

$T(x, t)$	is the temperature of the body as a function of position and time;
T_a	is the ambient temperature;
T_0	is the initial temperature of the body;
$\theta(\xi, \tau) = T(x, t)/T_a$	is the relative temperature;
$\theta_s(\tau)$, $\theta_c(\tau)$	are the relative temperatures of the surface and center of body;
$\xi = x/R$	is the dimensionless coordinate;
$\tau = at/R^2$	is the dimensionless time;
2R	is the thickness of plate or diameter of cylinder or sphere;
τ_0	is the duration of inertial heating;
$\beta(\tau)$	is the depth of heating zone;
a	is the thermal diffusivity;
λ	is the thermal conductivity;
$\alpha(\tau)$	is the heat-transfer coefficient;
$Bi(\tau) = \alpha(\tau)R/\lambda$	is the Biot number;
m	is the form factor of body, equal to 0 for a plate, 1 for a cylinder, and 2 for a sphere.

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